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Note on the algebra of screening currents for the quantum-deformed W-algebra

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Abstract. With slight modifications in the zero modes contributions, the positive and negative screening currents for the quantum deformed W-algebra $\mathcal{W}_{q,p}(g)$ can be put together to form a single algebra which can be regarded as an elliptic deformation of the universal enveloping algebra of $\hat{g}$, where g is any classical simply laced Lie algebra.

Recently, various deformations of the classical and quantum Virasoro and W-algebras have received considerable interests. Frenkel and Reshetikhin [6] first introduced the Poisson algebras $\mathcal{W}_q(g)$, which are q-deformation of classical W-algebras. Later Shiraishi *et al* [13] obtained a quantum version of the algebra $\mathcal{W}_q(sl_2)$, which is a noncommutative algebra depending on two parameters, p and q. Awata *et al* [1] and Feigin and Frenkel [4] independently extended this result to the general case, i.e. quantum deformed W-algebras $\mathcal{W}_{q,p}(g)$, where g is any classical semisimple Lie algebra. All these algebras were obtained together with their respective bosonic Fock space representations. Similar considerations, with respect to the Yangian deformation, were also carried out and have led to $\hbar$-deformed Virasoro algebra [3] and quantum $(\xi, \hbar)$-deformed W-algebras [8].

In [4] they also obtained the screening currents for the algebra $\mathcal{W}_{q,p}(g)$ and found the elliptic relations between them. They noticed that the relations for the positive (resp. negative) screening currents form an elliptic deformation for the loop algebra $\hat{n}_+$ (resp. $\hat{n}_-$) of the nilpotent subalgebra n_+ (resp. n_-) of g. However, they did not consider whether these two nilpotent elliptic algebras can be put together to form a unified elliptic algebra.

In this note we shall show that it is possible to combine the above nilpotent algebras into a single unified elliptic algebra if we introduce some new generating currents denoted by $H_i^{\pm}(z)$ and slightly modify the zero mode contributions in the bosonic representation of the screening currents (the modified 'screening currents'§ are denoted by $E_i(z)$ and $F_i(z)$ respectively whilst the original ones are denoted by $S_i^{\pm}(z)$). Unlike the unmodified screening currents, the modified currents $E_i(z)$, $F_i(z)$ and the newly introduced currents $H_i^{\pm}(z)$ do not commute with the W-algebra generating currents even up to total differences.

Let us start with a brief description of the results of [4] which are necessary for our discussion. First we consider the simple case of $g = sl_N$. By definition, the algebra

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§ Strictly speaking, the modified currents are no longer screening currents of the deformed W-algebra and hence we here use the quotation marks.

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$\mathcal{W}_{q,p}(sl_N)$ is generated by the Fourier coefficients of the currents $T_1(z), \ldots, T_{N-1}(z)$, which, in the free field realization, obey the quantum deformed Miura transformation

$$D_{p^{-1}}^N - T_1(z)D_{p^{-1}}^{N-1} + T_2(z)D_{p^{-1}}^{N-2} - \cdots + (-1)^{N-1}T_{N-1}(z)D_{p^{-1}} + (-1)^N$$
$$= \,:(D_{p^{-1}} - \Lambda_1(z))(D_{p^{-1}} - \Lambda_2(zp))\cdots(D_{p^{-1}} - \Lambda_N(zp^{N-1})):$$

where $\Lambda_i, i = 1, \ldots, N$ are generating series of a Heisenberg algebra, $[D_a f](x) = f(xa)$. The screening currents $S_i^{\pm}(z)$ are solutions of the difference equations

$$D_q S_i^+(z) = p^{-1} : \Lambda_{i+1}(zp^{i/2})\Lambda_i(zp^{i/2})^{-1} S_i^+(z) :$$
$$D_{p/q} S_i^-(z) = p^{-1} : \Lambda_{i+1}(zp^{i/2})\Lambda_i(zp^{i/2})^{-1} S_i^-(z) : .$$

From the above formulae one can show that the screening currents $S_i^+(z)$ and $S_i^-(z)$ commute with the W-algebra generating currents $T_i(z)$ up to a total difference. The case of other simply laced g of rank $N-1$ can be understood in a similar fashion with an appropriate modification in the form of a Miura transformation.

Let p, q be two generic parameters such that $|p/q| < 1$. We introduce a third parameter β using the relation $p = q^{1-\beta}$. For general simply laced Lie algebra g with the Cartan matrix (A_{ij}), we introduce the Heisenberg algebra [4] $\mathcal{H}_{q,p}(g)$ with generators $a_i[n]$, $i = 1, \ldots, N-1$; $n \in Z$ and Q_i, $i = 1, \ldots, N-1$ and relations

$$[a_i[n], a_j[m]] = \frac{1}{n}\frac{(1-q^n)(p^{A_{ij}n/2} - p^{-A_{ij}n/2})(1-(p/q)^n)}{1-p^n}\delta_{n,-m}$$
$$[a_i[n], Q_j] = A_{ij}\beta\delta_{n,0}.$$

Let

$$s_i^+[m] = \frac{a_i[m]}{q^{-m}-1} \qquad m \neq 0 \qquad s_i^+[0] = a_i[0]$$
$$s_i^-[m] = -\frac{a_i[m]}{(q/p)^m - 1} \qquad m \neq 0 \qquad s_i^-[0] = a_i[0]/\beta.$$

Then the screening currents $S_i^+(z)$ and $S_i^-(z)$ can be realized in the Fock space of $\mathcal{H}_{q,p}(g)$ as [4]

$$S_i^+(z) = \mathrm{e}^{Q_i} z^{s_i^+[0]} : \exp\left(\sum_{m\neq 0} s_i^+[m] z^{-m}\right) : \tag{1}$$

$$S_i^-(z) = \mathrm{e}^{-Q_i/\beta} z^{-s_i^-[0]} : \exp\left(-\sum_{m\neq 0} s_i^-[m] z^{-m}\right) : . \tag{2}$$

Using these bosonic expressions, Feigin and Frenkel [4] arrived at the following elliptic relations for the screening currents $S_i^+(z)$ and $S_i^-(z)$,

$$S_i^+(z)S_j^+(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z}\right)^{A_{ij}-A_{ij}\beta-1}\frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}\right)}S_j^+(w)S_i^+(z) \tag{3}$$

$$S_i^-(z)S_j^-(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z}\right)^{A_{ij}-A_{ij}/\beta-1}\frac{\theta_{p/q}\left(\frac{w}{z}p^{A_{ij}/2}\right)}{\theta_{p/q}\left(\frac{z}{w}p^{A_{ij}/2}\right)}S_j^-(w)S_i^-(z) \tag{4}$$

where as usual,

$$\theta_a(x) = (x|a)_\infty(ax^{-1}|a)_\infty(a|a)_\infty \qquad (x|a)\infty \equiv \prod_{n=0}^{\infty}(1-xa^n)$$

and equations (3) and (4) are to be understood in the sense of analytical continuation. Note that the function $\theta_a(x)$ is an elliptic function with multiplicative periods a and $e^{2\pi i}$,

$$\theta_a(ax) = -x^{-1}\theta_a(x) \qquad \theta_a(xe^{2\pi i}) = \theta_a(x).$$

In [4], Feigin and Frenkel also obtained the cross relations between $S_i^+(z)$ and $S_i^-(z)$ as normal ordered relations,

$$S_i^+(z)S_i^-(w) = \frac{1}{(z-wq)(z-wp^{-1}q)} : S_i^+(z)S_i^-(w) : \tag{5}$$

$$S_i^+(z)S_j^-(w) = (z-wp^{-1/2}q) : S_i^+(z)S_j^-(w) : \qquad A_{ij} = -1 \tag{6}$$

$$S_i^+(z)S_j^-(w) =: S_i^+(z)S_j^-(w) : \qquad A_{ij} = 0. \tag{7}$$

The reversed relation can also be obtained straightforwardly,

$$S_i^-(w)S_i^+(z) = \frac{1}{(w-zq^{-1})(w-zpq^{-1})} : S_i^+(z)S_i^-(w) : \tag{8}$$

$$S_j^-(w)S_i^+(z) = (w-zp^{1/2}q^{-1}) : S_i^+(z)S_j^-(w) : \qquad A_{ij} = -1 \tag{9}$$

$$S_j^-(w)S_i^+(z) =: S_i^+(z)S_j^-(w) : \qquad A_{ij} = 0. \tag{10}$$

The primary motivation of this work was to combine the algebra of positive and negative screening currents $S_i^{\pm}(z)$ into a single unified algebra. For this the relation between $S_i^+(z)$ and $S_j^-(w)$ have to be closed in the sense of commutator algebra. However, as can be easily shown from equations (5)–(10), this is impossible if we use the original form of the screening currents, because on the right-hand side of the commutation relation (here $\simeq$ implies 'equals up to regular terms')

$$[S_i^+(z), S_j^-(w)] \simeq \delta_{ij}\left(\frac{1}{(z-wq)(z-wp^{-1}q)} - \frac{1}{(w-zq^{-1})(w-zpq^{-1})}\right) : S_i^+(z)S_j^-(w) :$$

the operator $: S_i^+(z)S_j^-(w) :$ is bilocalized and the rational expression $\frac{1}{(z-wq)(z-wp^{-1}q)} - \frac{1}{(w-zq^{-1})(w-zpq^{-1})}$ cannot be rewritten as the sum of δ-functions. This imply that the commutator $[S_i^+(z), S_j^-(w)]$ does not close over the space of operators depending only on one spectral parameter.

The way around this difficulty is to modify the screening currents slightly so that the commutator of the modified 'screening currents' indeed close over the space of operators depending on a single parameter.

The modified 'screening currents' read

$$E_i(z) = e^{Q_i}(z(p/q)^{1/2})^{P_i} : \exp\left(\sum_{m\neq 0} s_i^+[m]z^{-m}\right) : \tag{11}$$

$$F_i(z) = e^{-Q_i}(zq^{1/2})^{-P_i} : \exp\left(-\sum_{m\neq 0} s_i^-[m]z^{-m}\right) : \tag{12}$$

where $P_i = s_i^-[0]$, which satisfy the relation

$$[P_i, Q_j] = A_{ij}.$$

After the above modification, we can easily show that the relations for $E_i(z)$ and $F_i(z)$ read

$$E_i(z)E_j(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z}\right)^{-1}\frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}\right)}E_j(w)E_i(z) \tag{13}$$

$$F_i(z)F_j(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z}\right)^{-1}\frac{\theta_{p/q}\left(\frac{w}{z}p^{A_{ij}/2}\right)}{\theta_{p/q}\left(\frac{z}{w}p^{A_{ij}/2}\right)}F_j(w)F_i(z). \tag{14}$$

Moreover, equations (5)–(10) will be turned into

$$E_i(z)F_i(w) = \frac{1}{(z(p/q)^{1/2})^2\left(1-\frac{wq}{z}\right)\left(1-\frac{wp^{-1}q}{z}\right)} : E_i(z)F_i(w) : \tag{15}$$

$$E_i(z)E_j(w) = (z(p/q)^{1/2})\left(1-\frac{w}{z}p^{-1/2}q\right) : E_i(z)F_j(w) : \qquad A_{ij} = -1 \tag{16}$$

$$E_i(z)F_j(w) =: E_i(z)F_j(w) : \qquad A_{ij} = 0 \tag{17}$$

$$F_i(w)E_i(z) = \frac{1}{(wq^{1/2})^2\left(1-\frac{z}{wq}\right)\left(1-\frac{z}{wp^{-1}q}\right)} : E_i(z)F_i(w) : \tag{18}$$

$$F_j(w)E_i(z) = (wq^{1/2})\left(1-\frac{z}{w}p^{1/2}q^{-1}\right) : E_i(z)F_j(w) : \qquad A_{ij} = -1 \tag{19}$$

$$F_j(w)E_i(z) =: E_i(z)F_j(w) : \qquad A_{ij} = 0 \tag{20}$$

following from which we have the commutation relation

$$[E_i(z), F_j(w)] \simeq \frac{\delta_{ij}}{(p-1)zw}\left[\delta\left(\frac{w}{zq}\right)H_i^+(zq^{-1/2}) - \delta\left(\frac{w}{z(p/q)}\right)H_i^-(w(p/q)^{-1/2})\right] \tag{21}$$

where we have introduced the new generating currents $H_i^{\pm}(z)$

$$H_i^+(z) =: E_i(zq^{1/2})F_i(zq^{-1/2}) : \tag{22}$$

$$H_i^-(z) =: E_i(z(p/q)^{-1/2})F_i(z(p/q)^{1/2}) : \tag{23}$$

which play the role of Cartan subalgebra generators, and the δ-function is defined as

$$\delta(z) = \sum_{n\in Z} z^n \qquad f(z)\delta(z/w) = f(w)\delta(z/w).$$

From equations (13)–(21) we can further obtain the following commutation relations which should all be understood in the sense of analytical continuations,

$$H_i^{\pm}(z)H_j^{\pm}(w) = \left(\frac{w}{z}\right)^{-2}\frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}\right)\theta_{\tilde{q}}\left(\frac{w}{z}p^{A_{ij}/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}\right)\theta_{\tilde{q}}\left(\frac{z}{w}p^{A_{ij}/2}\right)}H_j^{\pm}(w)H_i^{\pm}(z) \tag{24}$$

$$H_i^+(z)H_j^-(w) = \left(\frac{w}{z}\right)^{-2}\frac{\theta_q\left(\frac{w}{z}p^{(A_{ij}}p^{-1/2}\right)\theta_{\tilde{q}}\left(\frac{w}{z}p^{(A_{ij}}p^{1/2}\right)}{\theta_q\left(\frac{z}{w}p^{(A_{ij}}p^{1/2}\right)\theta_{\tilde{q}}\left(\frac{z}{w}p^{(A_{ij}}p^{-1/2}\right)}H_j^-(w)H_i^+(z) \tag{25}$$

$$H_i^+(z)E_j(w) = (-1)^{A_{ij}-1}\left(\frac{w}{zq^{1/2}}\right)^{-1}\frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}q^{-1/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}q^{1/2}\right)}E_j(w)H_i^+(z) \tag{26}$$

$$H_i^-(z)E_j(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z(p/q)^{-1/2}}\right)^{-1}\frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}(p/q)^{1/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}(p/q)^{-1/2}\right)}E_j(w)H_i^-(z) \tag{27}$$

$$H_i^+(z)F_j(w) = (-1)^{A_{ij}-1}\left(\frac{w}{zq^{-1/2}}\right)^{-1}\frac{\theta_{p/q}\left(\frac{w}{z}p^{A_{ij}/2}q^{1/2}\right)}{\theta_{p/q}\left(\frac{z}{w}p^{A_{ij}/2}q^{-1/2}\right)}F_j(w)H_i^+(z) \tag{28}$$

$$H_i^-(z)F_j(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z(p/q)^{1/2}}\right)^{-1}\frac{\theta_{p/q}\left(\frac{w}{z}p^{A_{ij}/2}(p/q)^{-1/2}\right)}{\theta_{p/q}\left(\frac{z}{w}p^{A_{ij}/2}(p/q)^{1/2}\right)}F_j(w)H_i^-(z). \tag{29}$$

For $g = sl_2$, equations (13), (14), (21), (24)–(29) already define an elliptic algebra which can be regarded as the $c = 1$ case of the following more general elliptic deformation of the universal enveloping algebra of the Kac–Moody algebra $\widehat{sl}_2$†,

$$H^{\pm}(z)H^{\pm}(w) = \left(\frac{w}{z}\right)^{-2} \frac{\theta_q\left(\frac{w}{z}p\right)\theta_{\tilde{q}}\left(\frac{w}{z}p\right)}{\theta_q\left(\frac{z}{w}p\right)\theta_{\tilde{q}}\left(\frac{z}{w}p\right)} H^{\pm}(w)H^{\pm}(z) \tag{30}$$

$$H^{+}(z)H^{-}(w) = \left(\frac{w}{z}\right)^{-2} \frac{\theta_q\left(\frac{w}{z}p^{(2-c)/2}\right)\theta_{\tilde{q}}\left(\frac{w}{z}p^{(2+c)/2}\right)}{\theta_q\left(\frac{z}{w}p^{(2+c)/2}\right)\theta_{\tilde{q}}\left(\frac{z}{w}p^{(2-c)/2}\right)} H^{-}(w)H^{+}(z) \tag{31}$$

$$H^{+}(z)E(w) = -\left(\frac{wq^{-c/2}}{z}\right)^{-1} \frac{\theta_q\left(\frac{w}{z}pq^{-c/2}\right)}{\theta_q\left(\frac{z}{w}pq^{c/2}\right)} E(w)H^{+}(z) \tag{32}$$

$$H^{-}(z)E(w) = -\left(\frac{w\tilde{q}^{c/2}}{z}\right)^{-1} \frac{\theta_q\left(\frac{w}{z}p\tilde{q}^{c/2}\right)}{\theta_q\left(\frac{z}{w}p\tilde{q}^{-c/2}\right)} E(w)H^{-}(z) \tag{33}$$

$$H^{+}(z)F(w) = -\left(\frac{wq^{c/2}}{z}\right)^{-1} \frac{\theta_{\tilde{q}}\left(\frac{w}{z}pq^{c/2}\right)}{\theta_{\tilde{q}}\left(\frac{z}{w}pq^{-c/2}\right)} F(w)H^{+}(z) \tag{34}$$

$$H^{-}(z)F(w) = -\left(\frac{w\tilde{q}^{-c/2}}{z}\right)^{-1} \frac{\theta_{\tilde{q}}\left(\frac{w}{z}p\tilde{q}^{-c/2}\right)}{\theta_{\tilde{q}}\left(\frac{z}{w}p\tilde{q}^{c/2}\right)} F(w)H^{-}(z) \tag{35}$$

$$E(z)E(w) = -\left(\frac{w}{z}\right)^{-1} \frac{\theta_q\left(\frac{w}{z}p\right)}{\theta_q\left(\frac{z}{w}p\right)} E(w)E(z) \tag{36}$$

$$F(z)F(w) = -\left(\frac{w}{z}\right)^{-1} \frac{\theta_{\tilde{q}}\left(\frac{w}{z}p\right)}{\theta_{\tilde{q}}\left(\frac{z}{w}p\right)} F(w)F(z) \tag{37}$$

$$[E(z), F(w)] = \frac{1}{(p-1)zw}\left[\delta\left(\frac{z}{wq^{c}}\right)H^{+}(zq^{-c/2}) - \delta\left(\frac{w}{z\tilde{q}^{c}}\right)H^{-}(w\tilde{q}^{-c/2})\right] \tag{38}$$

where c is the central charge and $\tilde{q}$ and q are connected by the relation

$$q\tilde{q} = p^{c}.$$

For general g, equations (13), (14), (21), (24)–(29) do not yet form a closed algebra because the cubic Serre-like relations for $E_i(z)$ (resp. $F_i(z)$) have not been supplemented. Such Serre-like relations can be explicitly obtained using results (36) and (37). The final closed algebra has the following generating relations,

$$H_i^{\pm}(z)H_j^{\pm}(w) = \left(\frac{w}{z}\right)^{-2} \frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}\right)\theta_{\tilde{q}}\left(\frac{w}{z}p^{A_{ij}/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}\right)\theta_{\tilde{q}}\left(\frac{z}{w}p^{A_{ij}/2}\right)} H_j^{\pm}(w)H_i^{\pm}(z) \tag{39}$$

$$H_i^{+}(z)H_j^{-}(w) = \left(\frac{w}{z}\right)^{-2} \frac{\theta_q\left(\frac{w}{z}p^{(A_{ij}-c)/2}\right)\theta_{\tilde{q}}\left(\frac{w}{z}p^{(A_{ij}+c)/2}\right)}{\theta_q\left(\frac{z}{w}p^{(A_{ij}+c)/2}\right)\theta_{\tilde{q}}\left(\frac{z}{w}p^{(A_{ij}-c)/2}\right)} H_j^{-}(w)H_i^{+}(z) \tag{40}$$

$$H_i^{+}(z)E_j(w) = (-1)^{A_{ij}-1}\left(\frac{wq^{-c/2}}{z}\right)^{-1} \frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}q^{-c/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}q^{c/2}\right)} E_j(w)H_i^{+}(z) \tag{41}$$

$$H_i^{-}(z)E_j(w) = (-1)^{A_{ij}-1}\left(\frac{w\tilde{q}^{c/2}}{z}\right)^{-1} \frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}\tilde{q}^{c/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}\tilde{q}^{-c/2}\right)} E_j(w)H_i^{-}(z) \tag{42}$$

$$H_i^{+}(z)F_j(w) = (-1)^{A_{ij}-1}\left(\frac{wq^{c/2}}{z}\right)^{-1} \frac{\theta_{\tilde{q}}\left(\frac{w}{z}p^{A_{ij}/2}q^{c/2}\right)}{\theta_{\tilde{q}}\left(\frac{z}{w}p^{A_{ij}/2}q^{-c/2}\right)} F_j(w)H_i^{+}(z) \tag{43}$$

† We would like to thank H Awata for drawing our attention to [2], where the $c = 1$ form of the elliptic algebra in the case of $g = sl_2$ was obtained independently.

$$H_i^-(z)F_j(w) = (-1)^{A_{ij}-1}\left(\frac{w\tilde{q}^{-c/2}}{z}\right)^{-1}\frac{\theta_{\tilde{q}}\left(\frac{w}{z}p^{A_{ij}/2}\tilde{q}^{-c/2}\right)}{\theta_{\tilde{q}}\left(\frac{z}{w}p^{A_{ij}/2}\tilde{q}^{c/2}\right)}F_j(w)H_i^-(z) \tag{44}$$

$$E_i(z)E_j(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z}\right)^{-1}\frac{\theta_q\left(\frac{w}{z}p^{A_{ij}/2}\right)}{\theta_q\left(\frac{z}{w}p^{A_{ij}/2}\right)}E_j(w)E_i(z) \tag{45}$$

$$F_i(z)F_j(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z}\right)^{-1}\frac{\theta_{\tilde{q}}\left(\frac{w}{z}p^{A_{ij}/2}\right)}{\theta_{\tilde{q}}\left(\frac{z}{w}p^{A_{ij}/2}\right)}F_j(w)F_i(z) \tag{46}$$

$$[E_i(z), F_j(w)] = \frac{\delta_{ij}}{(p-1)zw}\left[\delta\left(\frac{z}{wq^c}\right)H^+(zq^{-c/2}) - \delta\left(\frac{w}{z\tilde{q}^c}\right)H^-(w\tilde{q}^{-c/2})\right] \tag{47}$$

$$E_i(z_1)E_i(z_2)E_j(w) - f_{ij}(z_1/w, z_2/w)E_i(z_1)E_j(w)E_i(z_2) + E_j(w)E_i(z_1)E_i(z_2)$$
$$+(\text{replacement } z_1 \leftrightarrow z_2) = 0 \qquad A_{ij} = -1 \tag{48}$$

$$F_i(z_1)F_i(z_2)F_j(w) - g_{ij}(z_1/w, z_2/w)F_i(z_1)F_j(w)F_i(z_2) + F_j(w)F_i(z_1)F_i(z_2)$$
$$+(\text{replacement } z_1 \leftrightarrow z_2) = 0 \qquad A_{ij} = -1 \tag{49}$$

where

$$f_{ij}(z_1/w, z_2/w) = \frac{\left(\psi_{ii}^{(q)}\left(\frac{z_2}{z_1}\right)+1\right)\left(\psi_{ij}^{(q)}\left(\frac{w}{z_1}\right)\psi_{ij}^{(q)}\left(\frac{w}{z_2}\right)+1\right)}{\psi_{ij}^{(q)}\left(\frac{w}{z_2}\right)+\psi_{ii}^{(q)}\left(\frac{z_2}{z_1}\right)\psi_{ij}^{(q)}\left(\frac{w}{z_1}\right)}$$

$$g_{ij}(z_1/w, z_2/w) = \frac{\left(\psi_{ii}^{(\tilde{q})}\left(\frac{z_2}{z_1}\right)+1\right)\left(\psi_{ij}^{(\tilde{q})}\left(\frac{w}{z_1}\right)\psi_{ij}^{(\tilde{q})}\left(\frac{w}{z_2}\right)+1\right)}{\psi_{ij}^{(\tilde{q})}\left(\frac{w}{z_2}\right)+\psi_{ii}^{(\tilde{q})}\left(\frac{z_2}{z_1}\right)\psi_{ij}^{(\tilde{q})}\left(\frac{w}{z_1}\right)}$$

in which

$$\psi_{ij}^{(q)}(x) = (-1)^{A_{ij}-1}(x)^{-1}\frac{\theta_q(xp^{A_{ij}/2})}{\theta_q(x^{-1}p^{A_{ij}/2})}$$

$$\psi_{ij}^{(\tilde{q})}(x) = (-1)^{A_{ij}-1}(x)^{-1}\frac{\theta_{\tilde{q}}(xp^{A_{ij}/2})}{\theta_{\tilde{q}}(x^{-1}p^{A_{ij}/2})}$$

are structure functions that appeared in the commutation relations between $E_i(z), E_j(w)$ and $F_i(z), F_j(w)$, and they admit the factorization property

$$\psi_{ij}^{(q)}(x) = \frac{\phi_{ij}^{(q)}(x)}{\phi_{ij}^{(q)}(x^{-1})} \qquad \psi_{ij}^{(\tilde{q})}(x) = \frac{\phi_{ij}^{(\tilde{q})}(x)}{\phi_{ij}^{(\tilde{q})}(x^{-1})}$$

where the functions $\phi_{ij}^{(q)}(x)$ and $\phi_{ij}^{(\tilde{q})}(x)$ are defined as follows

$$\phi_{ij}^{(q)}(x) = x^{-A_{ij}/2}\frac{\theta_q(xp^{A_{ij}/2})}{\theta_q(xq^{A_{ij}/2})} \qquad \phi_{ij}^{(\tilde{q})}(x) = x^{-A_{ij}/2}\frac{\theta_{\tilde{q}}(xp^{A_{ij}/2})}{\theta_{\tilde{q}}(x\tilde{q}^{A_{ij}/2})}.$$

These equations also imply that

$$\psi_{ij}^{(q)}(x)\psi_{ij}^{(q)}(x^{-1}) = 1 \qquad \psi_{ij}^{(\tilde{q})}(x)\psi_{ij}^{(\tilde{q})}(x^{-1}) = 1.$$

Equations (11), (12), (22), (23) give a level 1 representation of the algebra (39)–(49) on the Fock space of the Heisenberg algebra $\mathcal{H}_{q,p}(g)$.

To end this note, we would like to present some concluding remarks.

• With slight modifications to the zero mode contributions in the screening currents of the quantum-deformed W-algebras, we come to the new elliptic deformed Kac–Moody algebra described in (39)–(49). Such algebras should be regarded as being generated by the Fourier coefficients of the generating currents and are associative algebras with unit. However, unlike the usual quantum deformations of classical and affine Lie algebras, no Hopf algebraic or quasi-Hopf algebraic structures can be defined over this new kind of elliptic algebras.

• The algebra obtained in this note carries two deformation parameters p, q in contrast to the quantum group and Yangian algebras where only one deformation parameter q ($\hbar$) is present. Moreover, the relations for the 'positive' and 'negative' generating currents are deformed differently in the sense that the deformation parameters are different in these relations. The only known example of algebras of this kind before this note is the algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$, which are members of the so-called Hopf family of algebras [10, 9]. However, we are not able to define the structure of Hopf family over the present elliptic algebra. The relations between the new elliptic algebra and the Hopf family of algebras will be an interesting subject of further study.

• For quantum groups and Yangian algebras, there exist different realizations including the current realization, Reshetikhin–Semenov-Tian-Shansky realization [11] and the Drinfeld realizations. The elliptic algebra obtained in this note is only realized in the current realization. It remains a hard problem to find the other possible realizations, especially the realization which may have direct relation with the quantum Yang–Baxter relations.

• As mentioned earlier, the modification in the screening currents spoils the feature that they commute with the quantum-deformed W-algebra up to total differences. Therefore, the relationship between the resulting algebra and the quantum deformed W-algebra remains unclear. Presumably the quantum deformed W-algebras can be obtained from our elliptic algebra in terms of a (q, p)-deformed version of Hamiltonian reduction. It seems that this problem deserves to be studied in detail. Note that the q-difference version of Hamiltonian reduction has recently been carried out by Frenkel *et al* [7] and Semenov-Tian-Shansky and Sevostyanov [12].

• The elliptic algebra is obtained here only with the level $c = 1$ bosonic representation. It seems that much more effort should be paid towards the representation theory of this new kind of algebra, especially the representations with higher level and the irreducibility, the spinor representations and the level zero representations should be studied in detail. We do not know whether there exist any relationship between the level zero form of our algebra and the elliptic quantum groups proposed by Felder [5]. Hopefully there is some, but this problem can only be answered after a detailed study.

• Finally, the possible physical application of the new elliptic algebra should be studied, for example whether there exist any physical models bearing the new algebra as a symmetry. It is also interesting to mention that in conventional conformal field theories, the screening currents of the usual W-algebras also do not close into a single algebra. After similar modifications it is hopeful that they also form closed algebras.

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